Computing moment polytopes of tensors

with applications in

algebraic complexity

and quantum information

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together with

Matthias Christandl, Vladimir Lysikov, Harold Nieuwboer,

Michael Walter and Jeroen Zuiddam

Introduction

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The **moment polytope** of a tensor contains such information,

both geometric and asymptotic representation-theoretic

Introduction: relevance

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- non-commutative optimization through **scaling algorithms**, which optimize over such polytopes [Bürgisser-Franks-Garg-Oliveira-Walter-Wigderson, FOCS 2019]

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Only completely in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

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Obtaining **new examples** is an important step towards new **theoretic results**

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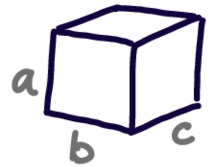
What are moment polytopes?

Moment polytopes: first definition

Fix a **tensor** $T \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$



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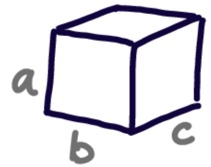
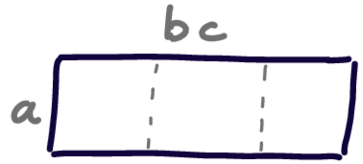


We can

- flatten to a **matrix** $T_1 \in \mathbb{C}^a \otimes (\mathbb{C}^b \otimes \mathbb{C}^c)$



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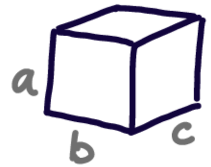

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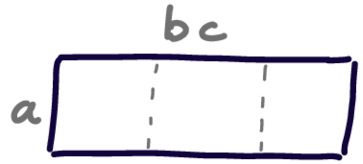
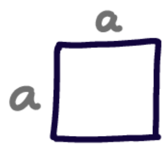


- consider $T_1 T_1^* \in \mathbb{C}^a \otimes \mathbb{C}^a$



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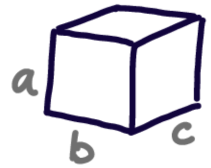

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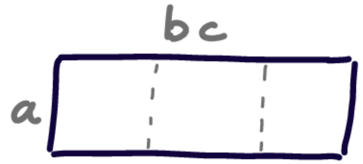

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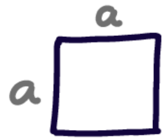

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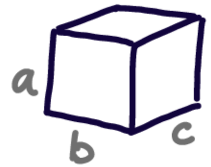

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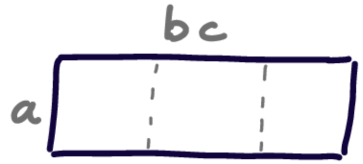

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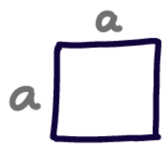

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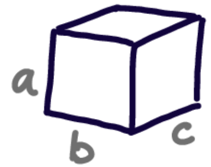

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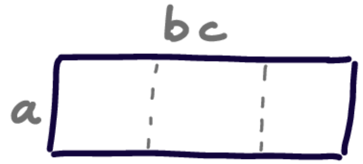

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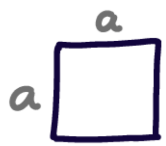

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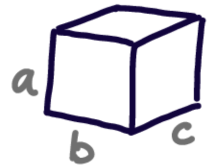

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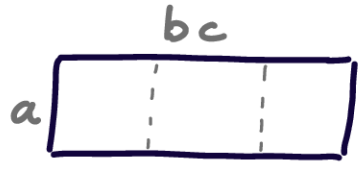

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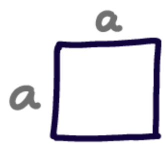

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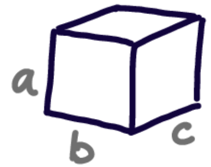

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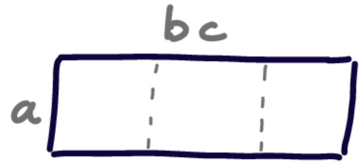

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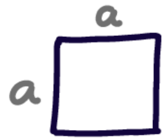

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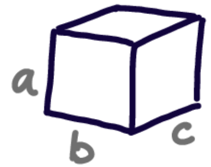

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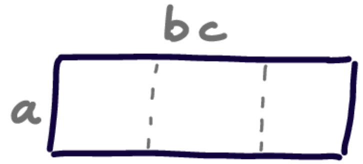

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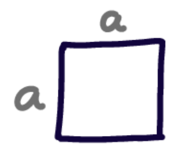

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$$\Big\{ \big(r_1(T'), r_2(T'), r_3(T') \big) \;\Big|\; T' \approx (\underset{\in \mathbb{C}^{a\times a}}{A} \otimes \underset{\in \mathbb{C}^{b\times b}}{B} \otimes \underset{\in \mathbb{C}^{c\times c}}{C})\, T \Big\}$$

$\subseteq \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$

$\hookrightarrow T' \in \overline{\{(A \otimes B \otimes C)T\}}$

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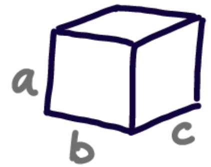

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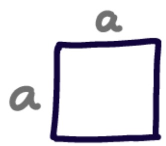

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Theorem [Mumford-Ness 1984]

$\Delta(T)$ is a rational (convex compact) polytope

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Theorem [Mumford-Ness 1984]

based on work in
- symplectic geometry
- invariant theory
- representation theory

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Computing moment polytopes

Our algorithm & Franz' description

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standard basis vector in $\mathbb{R}^a$

$$\mathrm{supp}(T) := \{ (e_i, e_j, e_k) \mid T_{i,j,k} \neq 0 \}$$

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$$D_n := \{ v \in \mathbb{R}^n \mid v_1 \geq \dots \geq v_n \}, \qquad D := D_a \times D_b \times D_c$$

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we need lots of optimizations to reach $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$

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Results : $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

Theorem

We know all moment polytopes in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

↳ there are 29 different ones

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- Classification due to [Nurmiev, Vinberg, Popov]
- Algorithm + some manual work
- Verify deterministically

Results: $\Delta(MM_n)$ is not maximal

Matrix multiplication tensors:

$$MM_n := \sum_{i,j,k=1}^{n} e_{(i,j)} \otimes e_{(j,k)} \otimes e_{(k,i)} \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2}$$

Unit tensors:

$$U_c := \sum_{i=1}^{c} e_i \otimes e_i \otimes e_i \in \mathbb{C}^{c} \otimes \mathbb{C}^{c} \otimes \mathbb{C}^{c}$$

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2x2 matrix multiplication

Our algorithm gave (probabilistically) that

$$\left(\left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right), \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0\right), \left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right) \right) \notin \Delta(MM_2)$$

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  → we obtain obstructions

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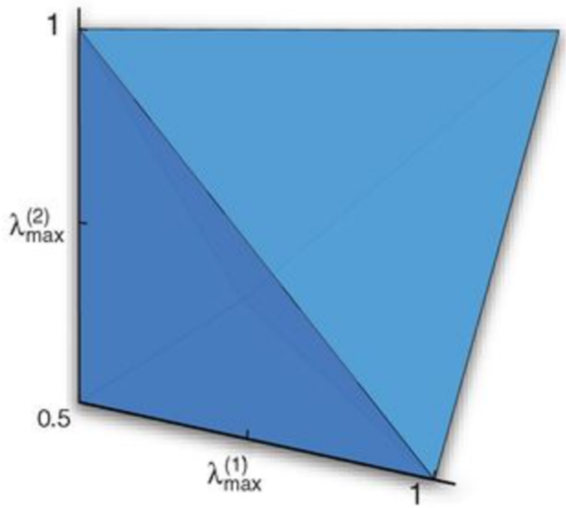
Thanks

Examples

$$\Delta\left(\sum_{i=1}^{2} e_i\otimes e_i\otimes e_i\right) = \mathrm{conv}\left\{\begin{array}{c} \overbrace{\quad\quad}^{\mathbb{R}^2}\ \overbrace{\quad\quad}^{\mathbb{R}^2}\ \overbrace{\quad\quad}^{\mathbb{R}^2} \\ (1,0,\ 1,0,\ 1,0) \\ (\tfrac12,\tfrac12,\ \tfrac12,\tfrac12,\ \tfrac12,\tfrac12) \\ (1,0,\ \tfrac12,\tfrac12,\ \tfrac12,\tfrac12) \\ (\tfrac12,\tfrac12,\ 1,0,\ \tfrac12,\tfrac12) \\ (\tfrac12,\tfrac12,\ \tfrac12,\tfrac12,\ 1,0) \end{array}\right\}$$

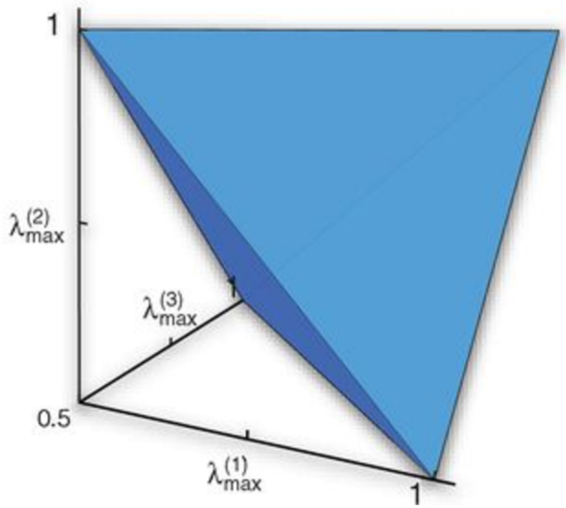
↳ unit tensor (order 2)



$$\Delta\left(\begin{array}{l} e_1\otimes e_1\otimes e_2 \\ +e_1\otimes e_2\otimes e_1 \\ +e_2\otimes e_1\otimes e_1 \end{array}\right) = \mathrm{conv}\left\{\begin{array}{c} (1,0,\ 1,0,\ 1,0) \\ (1,0,\ \tfrac12,\tfrac12,\ \tfrac12,\tfrac12) \\ (\tfrac12,\tfrac12,\ 1,0,\ \tfrac12,\tfrac12) \\ (\tfrac12,\tfrac12,\ \tfrac12,\tfrac12,\ 1,0) \end{array}\right\}$$

↳ W tensor

slices: $\begin{matrix}0&1\\1&0\end{matrix}$ $\quad$ $\begin{matrix}1&0\\0&0\end{matrix}$



Results: asymptotic restriction

Easy to see: $T \geq S \implies \Delta(S) \subseteq \Delta(T)$

↳ notation for $S = (A \otimes B \otimes C)T$

Asymptotic restriction:

Def: $T \gtrsim S$: $T^{\otimes n + o(n)} \geq S^{\otimes n} \quad \forall n$

Natural possibility:

$$T \gtrsim S \overset{?}{\implies} \Delta(S) \subseteq \Delta(T)$$

We show that **no**, $\not\Longrightarrow$

↳ by counterexample

Results: first explicit free tensor

T is **free** if some $T' \in (A \otimes B \otimes C)T$ has **free support**
$\hookrightarrow$ invertible

$$T'_{i,j,k} \neq 0 \Rightarrow T'_{i',j,k} = 0 \quad \forall i' \neq i$$ and for j, k as well.

- Free tensors play a role in Strassen's theory of asymptotic spectra

- Most tensors are **not free**, but none were known

We **construct** non-trivial non-free tensors in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$:

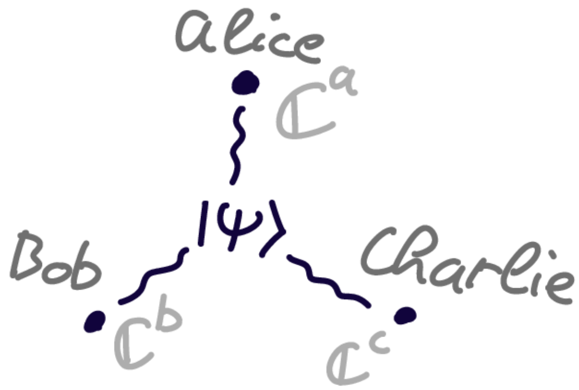
$$\left[\begin{array}{cc|cc|cc} & & & 1 & 1 & 0 \\ & 1 & 0 & 1 & & \\ 1 & 1 & & & & \end{array}\right], \quad \left[\begin{array}{ccc|ccc|ccc|ccc} & & & & & & & & 1 & 1 & 0 & 0 \\ & & & & & 1 & 0 & 1 & 0 & & & \\ & & 1 & 0 & 0 & 1 & & & & & & \\ 1 & 1 & 1 & & & & & & & & & \end{array}\right], \quad \text{etc.}$$

Moment polytopes: a quantum slide

$$\Delta(T) := \{ (r_1(T'), r_2(T'), r_3(T')) \mid T' \approx (A \otimes B \otimes C) T \}$$

$$|\varphi\rangle := \frac{T}{\|T\|} \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$$

is a **multipartite quantum state**



- $A \otimes B \otimes C$ is a SLOCC operation

  stochastic local operations & classical communication

  "operations that do not increase entanglement"

- Set $\rho = |\psi\rangle\langle\psi|$, then $r_1(T') = \mathrm{spec}(\rho_{alice})$

  $T' = |\psi\rangle$

$\Delta(|\varphi\rangle)$ characterizes the marginals (approximately) reachable under SLOCC

$\Longrightarrow$ it is called the **entanglement polytope** of $|\varphi\rangle$

Moment polytopes: representation theory

The group $GL_a \times GL_b \times GL_c$ acts on $(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)^{\otimes n}$

$$(A, B, C) \longmapsto (A \otimes B \otimes C)^{\otimes n}$$

Rep. theory tells us:

$$(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)^{\otimes n} = \bigoplus_{\lambda,\mu,\nu} V_{\lambda,\mu,\nu}$$

- $V_{\lambda,\mu,\nu}$ are **isotypic subspaces**
- labels are **partitions**: $\lambda \in \mathbb{N}^a \quad \sum_i \lambda_i = n \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_a$

Def.

projector onto

$$\Delta^{repr}(T) := \overline{\left\{ \left(\frac{\lambda}{n}, \frac{\mu}{n}, \frac{\nu}{n}\right) \;\middle|\; P_{\lambda,\mu,\nu}\, T^{\otimes n} \neq 0,\ n > 0 \right\}}$$

$$\subseteq \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$$

Computing entanglement polytopes is difficult

- Only known completely for $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 (\otimes \mathbb{C}^2)$
  + some sporadic examples

Aside: a decision problem

Problem: Given $T \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ & $p \in \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$, determine whether $p \in \Delta(T)$

- "Close" to (N)P & coNP
  ↳ bitsize of certificates is a problem
- Scaling methods can decide yes-instances in practice
- # vertices is typically not polynomial in dimension (a, b, c)

- Better understanding requires more examples

Some stats

$\in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

	Inequalities					Vertices	Runtimes	
Tensor	All	Not generic	Maxranks	Attainable	Final	Final	$\mathbb{Q}$	$\mathbb{F}_q$
$\langle 3 \rangle$	2845	2187	355	0	45	33	0.254	0.239
T_4	2845	2187	355	20	52	53	0.264	0.237
T_9	2845	2187	736	292	25	18	0.293	0.263
$\langle 4 \rangle$	8109383	7139405	1102518	0	270	328	-	20:10
$\mathrm{MM}_{2,2,2}$	8109383	7139405	1102518	1227	129	181	-	3:06

this step uses Gröbner bases

still going after 10 hours

$\in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$